$SL(2, Z)$ Duality and 4-Dimensional Noncommutative Theories

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ABSTRACT

We investigate how the four-dimensional noncommutative open string/Yang-Mills theory behaves under a general non-perturbative quantum $SL(2, Z)$ symmetry transformation. We discuss this by considering D3 branes in a constant background of axion, dilaton, and electric and magnetic fields (including both $\mathbf{E} \perp \mathbf{B}$ and $\mathbf{E} || \mathbf{B}$ cases) in the respective decoupling limit. We find that the value of axion, whether rational or irrational, determines the nature of the resulting theory under $SL(2, Z)$ as well as its properties such as the coupling constant and the number of noncommutative directions. In particular, a strongly coupled theory with an irrational value of axion can never be physically equivalent to a weakly coupled theory while this is usually true for a theory with a rational value of axion. A noncommutative Yang-Mills (NCYM) (resulting from D3 branes with pure magnetic flux) is physically equivalent to a noncommutative open string (NCOS) but if the value of axion is irrational, we also have noncommutative space-space directions in addition to the usual noncommutative space-time directions for NCOS. We also find in general that a NCOS cannot be physically equivalent to a NCYM but to another NCOS if the value of axion is irrational. We find another new decoupling limit for possible light-like NCYM whose $SL(2, Z)$ duality is a light-like ordinary Yang-Mills if the value of the axion is rational. Various related questions are also discussed.

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1 Introduction

There is a great surge of interest recently on noncommutativity along space-time directions in string/M theory [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. This non-commutativity implies a stringy uncertainty principle which appears as a special case of a fundamental one in string/M theory as advocated in [19]. A field theory defined on such a noncommutative geometry cannot be unitary [1, 6, 20] and therefore if there exists a decoupled theory on such geometry in string/M theory, it cannot be a field theory. In the context of D-branes with pure electric flux, it was shown in [2] that such a decoupled theory indeed exists and is a noncommutative open string (NCOS) living on the brane. In obtaining such a decoupled theory, the electric field must be close to its critical value such that it almost balances the original string tension. We therefore end up effectively with an almost tension-less string. Such a tiny string tension defines a new energy scale for the decoupled NCOS as well as the scale for the noncommutativity[1]. In this decoupling limit, the closed string coupling blows up while the coupling for the decoupled NCOS remains fixed and therefore is small in comparison with the infinitely large closed string coupling. This also implies that the transition of NCOS into closed strings cannot occur easily, indicating that the NCOS decouples from the bulk closed strings.

The same conclusion for the existence of NCOS was also obtained in [3] but from a rather different viewpoint. Consider a 4-dimensional noncommutative Yang-Mills (NCYM) which is the decoupled field theory of D3 branes in a purely magnetic field background. The decoupling limit for this theory requires the closed string coupling to scale to zero. When the gauge coupling for the NCYM is large, the natural way to deal with this theory is to go to its S-dual description. The authors in [3] asked: What is the S-dual of this NCYM? A worldvolume magnetic field is mapped to an electric field under S-duality[2]. Moreover, it was found that the decoupling limit in the S-dual theory exists only when the electric field attains a critical value. Also, in this case the original vanishing closed string coupling is

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[1]We can scale the original tension to infinity while keeping the new effective tension fixed.

[2]If one instead uses a B-field with a nonvanishing spatial component, one cannot reach the conclusion directly from the worldvolume point of view since a spatial B-field under S-duality turns into a spatial RR 2-form field. One often says that D3-branes (in general Dp-branes) with spatial nonvanishing B-field give rise to NCYM in the corresponding decoupling limit. From the field theory (or open string point of view) side, this is correct since all we need for the NCYM is the closed string metric, closed string coupling and the B-field as demonstrated in [21]. However, from the gravity description of D3-branes with spatial nontrivial B-field, we must also have D-strings present such that the resulting configuration is BPS. These D-strings require a nonvanishing space-time component of RR 2-form. Under S-duality, this nonvanishing component becomes a B-field which is needed for the space-time noncommutativity as demonstrated in [10, 12]. So this seems to indicate that the gravity description of NCYM contains more information than the field theory description. For example, in order to have the correct decoupling limit for NCOS, we have to perform the non-linear S-duality on the worldvolume while on the gravity side the S-duality is linear. This is the precise reason that we chose to work on a concrete gravity system of ((F, D1), D3) bound state for the decoupled NCOS with noncommutativity in both space-time and space-space directions in [12]. But in the prsent paper, we choose to work in the hard way, i.e. from the worldvolume side.

transformed to its inverse in the S-dual, therefore becoming infinitely large. So, according to what has been obtained in [2], the S-dual of NCYM is a NCOS.

The NCYM used in [3] arises as a decoupled theory of D3 branes with a pure magnetic flux. We can have decoupled NCYM from D3 branes with both electric and magnetic fields as discussed in [11, 12, 13] and further with nonvanishing axion which will be discussed in this paper[3]. We can also have decoupled NCOS from D3 branes in the presence of both electric and magnetic fields as discussed in [11, 12, 13, 22] and further with a nonvanishing axion which, for the $\mathbf{E}||\mathbf{B}$ case, has been discussed in [13, 22]. One may wonder if the S-dual or in general a $SL(2, Z)$ dual of NCYM (or NCOS) always give a NCOS (or NCYM). In [12], the present authors showed that the S-dual of NCYM does not in general give a NCOS. In [13] for the case $\mathbf{E}||\mathbf{B}$, it was shown that a NCOS is mapped to another NCOS under a $SL(2, Z)$ tranformation if the value of axion is irrational. Only for rational value of axion, a NCOS is physically equivalent to a NCYM.

We here intend to give a systematic study of the $SL(2, Z)$ dual of NCYM/NCOS which arises from D3 branes with nonvanishing axion and in the presence of both electric and magnetic fields in the respective decoupling limits. We will consider both of $\mathbf{E}||\mathbf{B}$ and $\mathbf{E}\perp\mathbf{B}$ cases. Our plan is as follows: In the following section, we give explicit $SL(2, Z)$ transformation rules for the worldvolume gauge fields and the generalized worldvolume gauge fields in the most general background. In section 3, we will calculate quantities relevant for decoupling limits for NCYM/NCOS in two versions related by $SL(2, Z)$ duality using Seiberg-Witten relations[21]. We consider both $\mathbf{E}\perp\mathbf{B}$ and $\mathbf{E}||\mathbf{B}$ cases. In section 4, we will discuss various decoupling limits for NCOS/NCYM and the $SL(2, Z)$ dual of NCYM/NCOS. We conclude this paper in section 5.

2   D3 Branes and Nonperturbative $SL(2, Z)$ Symmetry

Type IIB string theory is conjectured to have a nonperturbative quantum $SL(2, Z)$ symmetry. This symmetry implies that two Type IIB string theories related by an $SL(2, Z)$ transformation are physically equivalent. Among various dynamical objects in this theory, D3 branes play a special role in the sense that this object, like type IIB string itself, is invariant under the $SL(2, Z)$. For the purpose of this paper, we need to consider only the bosonic sector of the low energy effective action for a single D3 brane coupled to a most general background. This is described by a Born-Infeld type action [23, 24, 25] in Einstein-frame as

$$S_4 = -\frac{1}{(2\pi)^3 \alpha'^2} \int d^4x \sqrt{-\det(g^E + e^{-\phi/2}\mathcal{F})} + \frac{1}{(2\pi)^3 \alpha'^2} \int d^4x (C \wedge e^{\mathcal{F}})_4, \qquad (1)$$

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[3]While we were writing up this paper, we became aware a paper [22] in which the $SL(2, Z)$ duality of the gravity dual description of ((F, D1), D3) bound state, corresponding to $\mathbf{E}||\mathbf{B}$ case considered here, appeared on the net.

where $\mathcal{F}_2 = 2\pi\alpha' F_2 + B_2$ and

$$(C \wedge e^{\mathcal{F}})_4 = C_4 + C_2 \wedge \mathcal{F} + \frac{1}{2}C_0\mathcal{F}_2 \wedge \mathcal{F}_2. \tag{2}$$

In the above, the 2-form F_2 is the worldvolume $U(1)$ field strength, the worldvolume metric is the pullback of spacetime metric, B_2 is the pullback of spacetime NSNS 2-form potential and C_n is the pullback of spacetime RR n-form potential. In the following, we take $\mu, \nu = 0, 1, 2, 3$ as worldvolume indices and we will denote $C_0 = \chi$ for notational convenience.

For vanishing C_2 and B_2, it was shown in [26] that the equations of motion from the above action are just special forms of a more general action which possess a classical $SL(2, R)$ symmetry provided the dilaton and the axion parametrize the coset $SL(2, R)/SO(2)$. It was further shown in [23, 24] that with the inclusion of both B_2 and C_2, the equations of motion still have such $SL(2, R)$ symmetry provided the metric, dilaton, axion, B_2, C_2 and C_4 transform according to the rules determined from the type IIB supergravity, i.e.,

$$g_{\mu\nu}^E \to g_{\mu\nu}^E, \quad C_4 \to C_4, \quad \lambda \to \frac{a\lambda + b}{c\lambda + d},$$
$$\begin{pmatrix} B_2 \\ C_2 \end{pmatrix} \to \left(\Lambda^{-1}\right)^T \begin{pmatrix} B_2 \\ C_2 \end{pmatrix}, \tag{3}$$

where the complex scalar[4] $\lambda = \chi + ie^{-\phi}$, Λ is a 2×2 $SL(2, R)$ matrix defined as

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1, \tag{4}$$

and 'T' denotes the transpose of the matrix.

Our goal in this section is to express the transformed F_2 (or $\mathcal{F}_2$) under $SL(2, R)$ in terms of the original fields in a simple way. As is understood, the $SL(2, R)$ symmetry is manifest only on equations of motion and it rotates between equations of motion and Bianchi identities for F_2. Let us define a quantity $K^{\mu\nu}$ as

$$\sqrt{-\det g^E}\frac{K^{\mu\nu}}{2\pi} = \frac{\delta S_4}{\delta F_{\mu\nu}}. \tag{5}$$

Note that

$$\det\left(g^E + e^{-\phi/2}\mathcal{F}\right) = \left(\det g^E\right)\left(1 + \frac{1}{2}e^{-\phi}\mathcal{F}^2 - \frac{1}{16}e^{-2\phi}\left(\mathcal{F} \star \mathcal{F}\right)^2\right), \tag{6}$$

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[4]Alternatively, the dilaton and axion can be used to parameterize the coset $SL(2, R)/SO(2)$ as

$$\mathcal{M} = \begin{pmatrix} \chi^2 + e^{-2\phi} & \chi \\ \chi & 1 \end{pmatrix} e^{\phi}.$$

Then under $SL(2, R)$, $\mathcal{M} \to \Lambda\mathcal{M}\Lambda^T$ with $SL(2, R)$ matrix Λ defined in (4).

where $\star$ denotes the Hodge-dual on the brane. With the above, we have

$$2\pi\alpha' K_{\mu\nu} = -\frac{e^{-\phi}\mathcal{F}_{\mu\nu} - \frac{1}{4}e^{-2\phi}(\mathcal{F}\star\mathcal{F})(\star\mathcal{F})_{\mu\nu}}{\sqrt{1 + \frac{1}{2}e^{-\phi}\mathcal{F}^2 - \frac{1}{16}e^{-2\phi}(\mathcal{F}\star\mathcal{F})^2}} + (\star C)_{\mu\nu} + \chi(\star\mathcal{F})_{\mu\nu}. \tag{7}$$

With the above expression, the equation of motion for gauge potential A $(F_2 = dA)$ is

$$d\star K_2 = 0. \tag{8}$$

So combining with Bianchi identity $dF_2 = 0$, we have

$$d\begin{pmatrix} F_2 \\ \star K_2 \end{pmatrix} = 0. \tag{9}$$

Given any solution of the above equation for F_2 (K_2 is given through (7)), it appears that we could obtain another solution from this through a global $GL(2,R)$ rotation. But, since D3-branes appear as sources to the bulk gravity, the energy-momentum tensor due to this source must be kept invariant under this symmetry since the Einstein-frame metric is inert to this symmetry. Further the equations of motion for various potentials in the bulk spacetime should be transformed covariantly under this transformation when the D3 brane source is considered. The global symmetry for the bulk gravity therefore restricts us to have only a global classical $SL(2,R)$ rather than $GL(2,R)$ for the D3 brane. With the D3 brane as source, we can deduce from equations of motion for B_2 and C_2, that $(F_2, \star K_2)$ transform in the same way as (B_2, C_2) under $SL(2,R)$, i.e.,

$$\begin{pmatrix} F_2 \\ \star K_2 \end{pmatrix} \rightarrow \left(\Lambda^{-1}\right)^T \begin{pmatrix} F_2 \\ \star K_2 \end{pmatrix}. \tag{10}$$

We can define a generalized 2-form $\mathcal{K}_2$ as

$$\mathcal{K}_2 = 2\pi\alpha' \star K_2 + C_2, \tag{11}$$

analogous to $\mathcal{F} = 2\pi\alpha' F_2 + B_2$. Given the transformations for $(F_2, \star K_2)$ and (B_2, C_2) under $SL(2,R)$, we have

$$\begin{pmatrix} \mathcal{F}_2 \\ \mathcal{K}_2 \end{pmatrix} \rightarrow \left(\Lambda^{-1}\right)^T \begin{pmatrix} \mathcal{F}_2 \\ \mathcal{K}_2 \end{pmatrix}. \tag{12}$$

This is the key equation which we will use in the following section. With K_2 given in (7), (11) can be re-expressed as

$$-\mathcal{K}_2 = \frac{e^{-\phi}(\star\mathcal{F})_2 + \frac{1}{4}e^{-2\phi}(\mathcal{F}\star\mathcal{F})\mathcal{F}_2}{\sqrt{1 + \frac{1}{2}e^{-\phi}\mathcal{F}^2 - \frac{1}{16}e^{-2\phi}(\mathcal{F}\star\mathcal{F})^2}} + \chi\mathcal{F}_2. \tag{13}$$

The above equation implies the following constraint which generalizes the one given in [26] as

$$(\mathcal{F}_2, \mathcal{K}_2)\mathcal{M}\begin{pmatrix} \star\mathcal{F}_2 \\ \star\mathcal{K}_2 \end{pmatrix} = 0, \tag{14}$$

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which is useful for proving the invariance of the energy-momentum tensor.

In the following section, we will use the above equations (12) and (13) for calculating the open string metric and noncommutativity parameters from the Seiberg-Witten relations for both $\mathbf{E}||\mathbf{B}$ and $\mathbf{E} \perp \mathbf{B}$ cases.

3 Seiberg-Witten Setup

In this section, we calculate the effective open string metric and noncommutativity parameters for both $\mathbf{E}\perp\mathbf{B}$ and $\mathbf{E}||\mathbf{B}$ cases using Seiberg-Witten formulae[21]. The effective open string metric is

$$G_{\mu\nu} = g_{\mu\nu} - (\mathcal{F}\ g^{-1}\ \mathcal{F})_{\mu\nu}, \tag{15}$$

and the anti-symmetric noncommutativity parameter is

$$\Theta^{\mu\nu} = 2\pi\alpha' \left(\frac{1}{g+\mathcal{F}}\right)^{\mu\nu}_A, \tag{16}$$

where 'A' denotes the antisymmetric part. The effective open string coupling G_s is related to the closed string coupling $g_s = e^\phi$ through the following relation:

$$G_s = g_s \left(\frac{detG}{det(g+\mathcal{F})}\right)^{1/2} \tag{17}$$

As usual, we assume in the following that $\mathcal{F}_2$ is entirely given by the worldvolume field F_2 and set the NSNS B_2 to zero. In other words, we trade NSNS B_2 for the worldvolume F_2 through a gauge transformation. For either $\mathbf{E}\perp\mathbf{B}$ or $\mathbf{E}\,||\mathbf{B}$ case, we calculate the above open string quantities from the relevant closed quantities and the worldvolume $\mathcal{F}$ in two versions related by $SL(2,Z)$-duality. From now on, we limit ourselves to the non-perturbative quantum $SL(2,Z)$ symmetry rather than the classical $SL(2,R)$. In other words, we consider physically equivalent theories related by $SL(2,Z)$. For convenience, let us write down the transformed $e^{\hat\phi}$, $\hat\chi$, $\hat{\mathcal{F}}_2$ and $\hat{\mathcal{K}}_2$ in terms of the corresponding original fields and integral $SL(2,Z)$ elements a,b,c,d which satisfy $ad - bc = 1$ as

$$e^{\hat\phi} = e^\phi|c\lambda + d|^2, \quad \hat\chi = \frac{ac(\chi^2 + e^{-2\phi}) + (ad+bc)\chi + bd}{|c\lambda+d|^2},$$

$$\hat{\mathcal{F}}_2 = d\mathcal{F}_2 - c\mathcal{K}_2, \quad \hat{\mathcal{K}}_2 = -b\mathcal{F}_2 + a\mathcal{K}_2. \tag{18}$$

where $\mathcal{K}_2$ is given by (13). The string metric is defined as $g_{\mu\nu} = e^{\phi/2}g^E_{\mu\nu}$ and so we have

$$\hat{g}_{\mu\nu} = g_{\mu\nu}|c\lambda + d|. \tag{19}$$

We always denote the corresponding quantities in the $SL(2,Z)$ dual with 'hat' over the letters as indicated above. Let us begin with the $\mathbf{E}\perp\mathbf{B}$ case first.

3.1 E⊥B Case

Our starting point is to choose constant $F_{\mu\nu}$

$$\mathcal{F}_2 = 2\pi\alpha' \begin{pmatrix} 0 & E & 0 & 0 \\ -E & 0 & B & 0 \\ 0 & -B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{20}$$

and the constant closed string metric in string-frame as $g_{\mu\nu} = \text{diag}(-g_0, g_1, g_2, g_3)$ with g_0, g_1, g_2, g_3 all positive fixed parameters.

Using (15), we have the open string metric as

$$G_{\mu\nu} = \begin{pmatrix} -g_0(1 - \tilde{E}^2) & 0 & -\sqrt{g_0 g_2}\tilde{E}\tilde{B} & 0 \\ 0 & g_1(1 - \tilde{E}^2 + \tilde{B}^2) & 0 & 0 \\ -\sqrt{g_0 g_2}\tilde{E}\tilde{B} & 0 & g_2(1 + \tilde{B}^2) & 0 \\ 0 & 0 & 0 & g_3 \end{pmatrix}, \tag{21}$$

and using (16) the noncommutativity parameter as

$$\Theta^{\mu\nu} = \frac{1}{1 - \tilde{E}^2 + \tilde{B}^2} \begin{pmatrix} 0 & \tilde{E}/E_0 & 0 & 0 \\ -\tilde{E}/E_0 & 0 & -\tilde{B}/B_0 & 0 \\ 0 & \tilde{B}/B_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{22}$$

The open string coupling can be obtained from eq.(17) as,

$$G_s = g_s \left(1 - \tilde{E}^2 + \tilde{B}^2\right)^{1/2}, \tag{23}$$

which implies that the critical field in this case is $(1 + \tilde{B}^2)^{1/2}$. In the above, we have defined

$$\tilde{E} = \frac{E}{E_0}, \quad \tilde{B} = \frac{B}{B_0}, \tag{24}$$

where the parameters $E_0 = \sqrt{g_0 g_1}/(2\pi\alpha')$ and $B_0 = \sqrt{g_1 g_2}/(2\pi\alpha')$.

One might think that the **E⊥B** case is simpler than the **E∥B** case. On the contrary, it is a bit more complicated in both the decoupling limits (NCYM and NCOS) which will be studied in the following section and the $SL(2, Z)$ dual formulation. Let us derive the open string metric, the noncommutativity parameter and the open string coupling in the $SL(2, Z)$ dual. In order to calculate these quantities, we have to express $\hat{\mathcal{F}}_2$ in terms of relevant quanties in the original version. For constant $F_{\mu\nu}$, the equation of motion is satisfied and so we can use the duality relation to calculate $\hat{\mathcal{F}}_2$. In doing so, we first need to calculate $\mathcal{K}_2$ from (13). Thus we find,

$$\mathcal{K}_{\mu\nu} = -\begin{pmatrix} 0 & \sqrt{g_0 g_1}\tilde{E}\chi & 0 & -\sqrt{g_0 g_3}\tilde{B}/G_s \\ -\sqrt{g_0 g_1}\tilde{E}\chi & 0 & \sqrt{g_1 g_2}\tilde{B}\chi & 0 \\ 0 & -\sqrt{g_1 g_2}\tilde{B}\chi & 0 & \sqrt{g_2 g_3}\tilde{E}/G_s \\ \sqrt{g_0 g_3}\tilde{B}/G_s & 0 & -\sqrt{g_2 g_3}\tilde{E}/G_s & 0 \end{pmatrix}, \tag{25}$$

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where we have used $g_{\mu\nu} = g_s^{1/2} g_{\mu\nu}^E$ and the relation between G_s and g_s given in (23) as well as the definitions for $\tilde{E}$ and $\tilde{B}$ given above. With the above and $\hat{\mathcal{F}}_2 = d\,\mathcal{F}_2 - c\,\mathcal{K}_2$, we have

$$
\hat{\mathcal{F}}_{\mu\nu} = \begin{pmatrix}
0 & \sqrt{g_0 g_1}\,\tilde{E}(c\chi + d) & 0 & -\sqrt{g_0 g_3}\,c\tilde{B}/G_s \\
-\sqrt{g_0 g_1}\,\tilde{E}(c\chi + d) & 0 & \sqrt{g_1 g_2}\,\tilde{B}(c\chi + d) & 0 \\
0 & -\sqrt{g_1 g_2}\,\tilde{B}(c\chi + d) & 0 & \sqrt{g_2 g_3}\,c\tilde{E}/G_s \\
\sqrt{g_0 g_3}\,c\tilde{B}/G_s & 0 & -\sqrt{g_2 g_3}\,c\tilde{E}/G_s & 0
\end{pmatrix}
\tag{26}
$$

We notice from the expression of $\hat{\mathcal{F}}_{\mu\nu}$ in (26) that we now have additional electric and magnetic fields pointing along negative x^3-direction and x^1-direction, respectively, in the $SL(2,Z)$ dual even though we originally had only electric field pointing along x^1-direction and magnetic field pointing along x^3-direction. This implies that we may have more noncommutative directions in the $SL(2,Z)$ dual. This differs from $\mathbf{E}||\mathbf{B}$ case as we will see.

Using again Seiberg-Witten relations and after some tedious calculations, we find the new open string metric to have the form

$$
\hat{G}_{\mu\nu} = \frac{|cS + d|^2}{|c\lambda + d|} G_{\mu\nu},
\tag{27}
$$

and the noncommutativity parameters take the form,

$$
\hat{\Theta}^{01} = \frac{(c\chi + d)}{|cS + d|^2}\Theta^{01}, \quad \hat{\Theta}^{03} = -\frac{2\pi\alpha'}{\sqrt{g_0 g_3}}\frac{c\tilde{B}/G_s}{|cS + d|^2},
$$

$$
\hat{\Theta}^{12} = \frac{(c\chi + d)}{|cS + d|^2}\Theta^{12}, \quad \hat{\Theta}^{23} = -\frac{2\pi\alpha'}{\sqrt{g_2 g_3}}\frac{c\tilde{E}/G_s}{|cS + d|^2},
\tag{28}
$$

where $\lambda = \chi + i/g_s$ defined before, $S = \chi + i/G_S$ (since $\mathbf{E}\cdot\mathbf{B} = 0$ here), and $G_{\mu\nu}$, Θ^{01} and Θ^{12} are the original open string metric, noncommutative parameters given in (21) and (22), respectively. The open string coupling here is related to the original open string coupling as

$$
\hat{G}_s = |cS + d|^2 G_s.
\tag{29}
$$

3.2 E || B Case

We now take

$$
\mathcal{F} = 2\pi\alpha' \begin{pmatrix}
0 & E & 0 & 0 \\
-E & 0 & 0 & 0 \\
0 & 0 & 0 & B \\
0 & 0 & -B & 0
\end{pmatrix},
\tag{30}
$$

and the string frame metric $g_{\mu\nu} = \text{diag}(-g_1, g_1, g_2, g_2)$ where g_1, g_2 are positive parameters. Using Seiberg-Witten relations, we have the open string metric as

$$
G_{\mu\nu} = \begin{pmatrix}
-g_1(1 - \tilde{E}^2) & 0 & 0 & 0 \\
0 & g_1(1 - \tilde{E}^2) & 0 & 0 \\
0 & 0 & g_2(1 + \tilde{B}^2) & 0 \\
0 & 0 & 0 & g_2(1 + \tilde{B}^2)
\end{pmatrix},
\tag{31}
$$

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the noncommutativity parameters as,

$$\Theta^{01} = \frac{\tilde{E}}{E_c(1 - \tilde{E}^2)}, \quad \Theta^{23} = -\frac{\tilde{B}}{B_0(1 + \tilde{B}^2)}, \tag{32}$$

and the open string coupling as

$$G_s = g_s(1 - \tilde{E}^2)^{1/2}(1 + \tilde{B}^2)^{1/2}. \tag{33}$$

In the above, we have defined $\tilde{E} = E/E_c$ and $\tilde{B} = B/B_0$ with the critical field $E_c = g_1/(2\pi\alpha')$ and $B_0 = g_2/(2\pi\alpha')$.

We now calculate the relevant open string quantities in the $SL(2, Z)$ dual. To do so, we need to have $\hat{\mathcal{F}}_2$ and as before we calculate $\mathcal{K}_{\mu\nu}$ first. Using (13), we have

$$\mathcal{K}_{01} = -g_1 \left[\tilde{E}\chi - \tilde{B}(1 - \tilde{E}^2)/G_s \right], \quad \mathcal{K}_{23} = -g_2 \left[\tilde{B}\chi + \tilde{E}(1 + \tilde{B}^2)/G_s \right], \tag{34}$$

where we have used the relation between G_s and g_s given in (33), the definitions for $\tilde{E}$ and $\tilde{B}$ given earlier and $g_{\mu\nu} = g_s^{1/2}g_{\mu\nu}^E$. With the above, we have using eq.(18)

$$\hat{\mathcal{F}}_{01} = g_1 \left[\tilde{E}(c\chi + d) - c\tilde{B}(1 - \tilde{E}^2)/G_s \right], \quad \hat{\mathcal{F}}_{23} = g_2 \left[\tilde{B}(c\chi + d) + c\tilde{E}(1 + \tilde{B}^2)/G_s \right]. \tag{35}$$

Using Seiberg-Witten relations (15)-(17), we now have the open string metric

$$\hat{G}_{\mu\nu} = \frac{|cS + d|^2}{|c\lambda + d|}G_{\mu\nu}, \tag{36}$$

the noncommutativity parameters

$$\hat{\Theta}^{01} = \frac{(c\chi + d)\Theta^{01} - \frac{c\tilde{B}}{E_{cr}G_s}}{|cS + d|^2}, \quad \hat{\Theta}^{23} = \frac{(c\chi + d)\Theta^{23} - \frac{c\tilde{E}}{B_0 G_s}}{|cS + d|^2}, \tag{37}$$

and the open string coupling

$$\hat{G}_s = |cS + d|^2 G_s. \tag{38}$$

In the above, G_s, $\Theta^{\mu\nu}$ and $G_{\mu\nu}$ are the original open string coupling, noncommutativity parameters and open string metric, respectively. We have now $S = \chi + \tilde{E}\tilde{B}/G_s + i/G_s$ for which **E** and **B** contribute since $\mathbf{E} \cdot \mathbf{B} \neq 0$.

4   Decoupling Limits and $SL(2, Z)$ Duality

We are now ready to discuss the decouping limits for NCYM/NCOS and the $SL(2, Z)$ duality for the underlying decoupled theory. Before we discuss the noncommutative theory, we would like to address one question: Can an ordinary theory become a noncommutative theory through $SL(2, Z)$ duality? Our examination gives negative answer. Let us point out that there is a general rule regarding whether we can map a strongly coupled theory to a weakly coupled one through a $SL(2, Z)$ transformation or not for any theory either ordianry or noncommutative. The rule is: for rational χ, we can have two physically equivalent theories which are strong-weak dual to each other while for irrational χ, we do not have this. We first discuss $\mathbf{E} \perp \mathbf{B}$ case and then $\mathbf{E}||\mathbf{B}$ case.

4.1 E ⊥ B Case

Let us begin with the decoupling limit for NCYM. To have a NCYM, we need to decouple not only the open string ending on the brane from the closed strings in the bulk but also the open string massive modes from the massless ones. So we need to send $\alpha' \to 0$. To have a sensible quantum theory, we need to fix the open string coupling and the open string metric in this limit. We also need to fix at least one nonvanishing spatial component of the noncommutative matrix. With these requirements and examining (21) and (22), we can naively have the following three limits:

- 1)

$$\alpha' \to 0, \quad g_1 = \left(\frac{\tilde{b}}{\alpha'}\right)^2, \quad \tilde{E}^2 = 1 + \tilde{B}^2 - \left(\frac{\alpha'}{\tilde{b}}\right)^2, \quad g_s = G_s \frac{\tilde{b}}{\alpha'} \tag{39}$$

with g_0, g_2, g_3 and $\tilde{B}$ fixed. For simplicity, we choose $g_0\tilde{B}^2 = 1$, $g_2(1 + \tilde{B}^2) = 1$ and $g_3 = 1$. So we have the metric

$$G_{\mu\nu} = \begin{pmatrix} 1 - g_0(\alpha'/\tilde{b})^2 & 0 & 1 - g_2(\alpha'/\tilde{b})^2/2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 - g_2(\alpha'/\tilde{b})^2/2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{40}$$

and the noncommutativity parameters[5]

$$\Theta^{01} = \Theta^{12} = 2\pi\tilde{b}|\tilde{B}|\sqrt{1 + \tilde{B}^2}. \tag{41}$$

- 2)

$$\alpha' \to 0, \quad \tilde{E} = -\tilde{B} = \left(\frac{\tilde{b}}{\alpha'}\right)^{1/2}, \quad g_0 = g_2 = \frac{\alpha'}{\tilde{b}}, \tag{42}$$

with g_1, g_3 and g_s fixed. We here choose $g_0/g_2 = 1$ just for simplicity but in general we need only g_0/g_2 fixed. For simplicity, we also choose $g_1 = g_3 = 1$. Now the open string metric is

$$G_{\mu\nu} = \begin{pmatrix} 1 - \frac{\alpha'}{\tilde{b}} & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 + \frac{\alpha'}{\tilde{b}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{43}$$

and the noncommutativity parameters are similar to those in 1) as

$$\Theta^{01} = \Theta^{12} = 2\pi\tilde{b}. \tag{44}$$

So 1) and 2) are quite similar except for the closed string coupling g_s. g_s blows up in the decoupling limit in 1) while it remains fixed in 2). This case corresponds to the light-like NCYM discussed in [14].

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[5]We choose $\tilde{B}$ to be negative for definiteness. For positive $\tilde{B}$, the discussion and the conclusion are basically the same.

- 3)

$$\alpha' \to 0, \quad g_1 = g_2 = \left(\frac{\alpha'}{\tilde{b}}\right)^2, \quad \tilde{B} = \frac{\tilde{b}}{\alpha'}, \quad g_s = G_s \frac{\alpha'}{\tilde{b}}, \tag{45}$$

with g_0, g_3 and $\tilde{E}$ fixed. Here we choose $g_1/g_2 = 1$ for simplicity but in general we only need this ratio to be fixed. For the present case, we do not have the $\tilde{E} \le 1$ requirement. It can be any fixed number or approaching zero. For simplicity we set $g_3 = 1$. This same decoupling limit for $\tilde{E} \le 1$ was discussed in [11]. With the above, we have the open string metric

$$G_{\mu\nu} = \begin{pmatrix} -g_0(1 - \tilde{E}^2) & 0 & -\sqrt{g_0}\tilde{E} & 0 \\ 0 & 1 & 0 & 0 \\ -\sqrt{g_0}\tilde{E} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{46}$$

and the nonvanishing noncommutativity parameter

$$\Theta^{12} = -2\pi\tilde{b}. \tag{47}$$

We point out that there are both space-time and space-space noncommutativities in 1) and 2) while there is only space-space noncommutativity in 3). The space-time non-commutativity arises because the electric field approaches the critical value in both cases. In general, one expects that the underlying decoupled theories are NCOS rather than a NCYM. At least for 2), the unitarity discussion given in [14] seems to indicate that the resulting theory is a light-like NCYM. Actually, 1) has the same structure as in 2). So we expect that we might also have a light-like NCYM in the limit of 1). The arguments given in [14] are: even though we have both Θ^{01} and Θ^{12} in appearance, if we choose light-like coordinates $x^\pm = (x^0 \pm x^2)/\sqrt{2}$, work in the (x^+, x^1, x^-, x^3)-system and take x^+ as the light-cone time, we then have only nonvanishing noncommutativity parameter Θ^{-1}, a space-space noncommutativity. In addition, the underlying theory is unitary, i.e., the inner product $p \circ p = -p_\mu \Theta^{\mu\rho} G_{\rho\sigma} \Theta^{\sigma\nu} p_\nu$ is never negative. Therefore, it appears that the underlying theory is a well-defined NCYM. Let us demonstrate this for both 1) and 2). We now express everything in the light-like coordinate (x^+, x^1, x^-, x^3)-system. Let us denote the corresponding cases as $1^{(lc)})$ and $2^{(lc)})$, respectively.

- $1^{(lc)})$: We now have the metric

$$G_{\mu\nu}^{(lc)} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{(\alpha'/\tilde{b})^2}{2g_0g_2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{48}$$

and the only nonvanishing noncommutativity parameter

$$\Theta^{(lc)-1} = -\Theta^{(lc)1-} = 2\sqrt{2}\pi\tilde{b}|\tilde{B}|\sqrt{1 + \tilde{B}^2}. \tag{49}$$

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One can check that indeed $p \circ p$ is non-negative as $p \circ p = -p_\mu \Theta^{\mu\rho} G_{\rho\sigma} \Theta^{\sigma\nu} p_\nu = (p_-)^2 (\Theta^{-1})^2 \geq 0$ when $\alpha' \to 0$ is taken.

- $2^{(lc)}$): We have the open string metric

$$
G_{\mu\nu}^{(lc)} = \begin{pmatrix} 2 & 0 & -\frac{\alpha'}{\tilde{b}} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\alpha'}{\tilde{b}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{50}
$$

and the nonvanishing noncommutativity parameter

$$
\Theta^{(lc)-1} = -\Theta^{(lc)1-} = 2\sqrt{2}\pi\tilde{b}. \tag{51}
$$

One can check again $p \circ p = (p_-)^2 (\Theta^{(lc)-1})^2 \geq 0$.

The above discussion seems to indicate that in terms of the light-like coordinates, $1^{(lc)}$) and $2^{(lc)}$) cases look no different from the case 3) above. According to the criterion given in [6, 14], each of the field theories is unitary and each has a space-space noncommutativity. We therefore should call the decoupled field theories in $1^{(lc)}$) and $2^{(lc)}$) as light-like NCYM.

Let us discuss the $SL(2, Z)$ duality for each of the NCYM. Let us denote the corresponding cases as $1^{(lc)}$)′, $2^{(lc)}$)′ and 3)′, respectively.

- $1^{(lc)}$)′ We need to consider irrational χ and rational χ separately. a) If χ is irrational, $|c\lambda + d| = (c\chi + d)$ since $g_s \to \infty$ in the decoupling limit 1). $|cS + d|^2 = (c\chi + d)^2 + c^2/G_s^2$ remains fixed. Using (27), (28), (29) and the decoupling limit in 1), we have

$$
\hat{G}_{\mu\nu}^{(lc)} = \frac{|cS + d|^2}{c\chi + d} G_{\mu\nu}^{(lc)}, \quad \hat{G}_s = |cS + d|^2 G_s, \quad \hat{\Theta}^{(lc)-1} = \frac{c\chi + d}{|cS + d|^2} \Theta^{(lc)-1}, \tag{52}
$$

where $G_{\mu\nu}^{(lc)}$, G_s and $\Theta^{(lc)-1}$ are the corresponding quantities in $1^{(lc)}$). So this theory looks similar to the original theory but it is always strongly coupled. The $SL(2, Z)$ duality here is not useful. b) If χ is rational, we can choose $c\chi + d = 0$. Now $|c\lambda + d| = |c|/g_s = |c|\alpha'/(G_s\tilde{b})$. $|cS + d|^2 = c^2/G_s^2$. We have now

$$
\hat{G}_{\mu\nu}^{(lc)} = \frac{\tilde{b}}{\alpha'} \frac{|c|]}{G_s} G_{\mu\nu}^{(lc)}, \quad \hat{G}_s = c^2/G_s, \tag{53}
$$

and all the noncommutativity parameters vanish. If the original light-like NCYM is strongly coupled, we end up with a physically equivalent weakly coupled theory defined on a commutative geometry. Also we have $p \circ p = 0$ since $\Theta^{(lc)\mu\nu}$ vanish. So it appears that we end up with a light-like OYM. We will comment on this later in this subsection.

- $2^{(lc)}$)′ We have also two cases: a) irrational χ and b) rational χ.

a)**Irrational** χ: Now $|c\lambda + d|$ is fixed since we have fixed g_s. So is $|cS + d|^2$. Therefore the metric $\hat{G}_{\mu\nu}^{(lc)}$ is essentially the same as the original metric $G_{\mu\nu}^{(lc)}$. We also have

$$\hat{\Theta}^{(lc)-1} = 2\sqrt{2}\pi\tilde{b}\frac{c\chi + d}{|cS + d|^2}, \quad \hat{\Theta}^{(lc)-3} = \frac{2\sqrt{2}\pi c\tilde{b}}{G_s|cS + d|^2}. \tag{54}$$

We again have only the space-space noncommutativity in the light-like coordinate system but we double the number of noncommutative pairs. This theory is also unitary since $p \circ p \geq 0$. We again end up with a light-like NCYM. However, this NCYM is strongly coupled regardless of the coupling of the original theory. So the $SL(2, Z)$ duality is not that useful except for increasing the noncommutative directions.

b)**Rational** χ: We can now choose $c\chi + d = 0$. Since g_s is fixed, we can obtain the corresponding quantities simply by setting $c\chi + d = 0$ in a). We then have $\hat{\Theta}^{(lc)-1} = 0$ and $\hat{G}_s = c^2/G_s$. So we end up with a weakly coupled light-like NCYM if the original light-like NCYM is strongly coupled. The number of noncommutative pairs remain the same but we change from $\Theta^{(lc)-1}$ to $\hat{\Theta}^{(lc)-3}$ by the $SL(2, Z)$ duality.

- 3)′ We have two cases: a) irrational χ and b) rational χ.
  a)**Irrational** χ: $|c\lambda + d| = |c|\tilde{b}/(\alpha' G_s)$. $|cS + d|^2$ remains fixed. We then have from (27) and (28)

$$\hat{G}_{\mu\nu} = \frac{\alpha'}{\tilde{b}}\frac{G_s}{|c|}|cS + d|^2 G_{\mu\nu}, \quad \hat{G}_s = |cS + d|^2 G_s,$$

$$\hat{\Theta}^{03} = -\frac{2\pi c\tilde{b}}{\sqrt{g_0}G_s|cS + d|^2}, \quad \hat{\Theta}^{12} = -\frac{2\pi\tilde{b}(c\chi + d)}{|cS + d|^2}, \quad \hat{\Theta}^{23} = -\frac{2\pi c\tilde{E}\tilde{b}}{G_s|cS + d|^2}, \tag{55}$$

Since $\alpha' G^{-1}$ is fixed, we therefore have NCOS. Depending on the value of $\tilde{E}$, we can have as many as three independent noncommutativity parameters. However, this theory is strongly coupled regardless of the original theory. So it is again not useful.
  b)**Rational** χ: For this case we can choose $c\chi + d = 0$. Now $|c\lambda + d|$ remains the same as the above since $g_s \to 0$ and $|cS + d|^2 = c^2/G_s^2$ still remains fixed. So we still have $\hat{G}_{\mu\nu} \sim \alpha' G_{\mu\nu}$ which implies that we still end up with a NCOS. But now $\hat{\Theta}^{12} = 0$. We have

$$\hat{\Theta}^{03} = -\frac{2\pi G_s\tilde{b}}{c\sqrt{g_0}}, \quad \hat{\Theta}^{23} = -2\pi G_s\tilde{E}\tilde{b}/c. \tag{56}$$

We then have a weakly coupled NCOS if the original NCYM is strongly coupled. So now the $SL(2, Z)$ duality is useful.

So far it appears that $1^{(lc)})$ and $2^{(lc)})$ differ from 3) in that the former are light-like NCYM while the latter is usual NCYM. Also their $SL(2, Z)$ dualities are quite different. The former give either light-like NCYM or OYM while the latter gives the usual NCOS. There is also a big difference regarding the closed string coupling g_s. The former have either g_s blowing up or fixed while the latter has vanishing g_s. In this aspect, these light-like NCYM are similar

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to the NCYM discussed in [12] whose S-duality gives an OYM which is not well-defined because of the singular metric and infinitely large open string coupling. We will discuss the $SL(2, Z)$ duality of this kind of NCYM in the following subsection. There is a difference regarding the open string coupling. The $SL(2, Z)$ duality of the light-like NCYM has a finite (maybe large) open string coupling while that of the above mentioned NCYM discussed in [12] has blowing up open string coupling. The reason for this is that we here consider $\mathbf{E} \perp \mathbf{B}$ case and $\mathbf{E}$ and $\mathbf{B}$ have no contribution to the quantity S. But for $\mathbf{E}||\mathbf{B}$ case, we do have $\tilde{E}\tilde{B}$ contribution to the S as indicated before which blows up in the decoupling limit for the above mentioned NCYM in [12].

Up to now we have avoided pointing out the underlying major difference between case $1^{(lc)})$ and $2^{(lc)})$ and case 3). The $\det G^{(lc)}_{\mu\nu} \sim \alpha'^2$ vanishes for the former two cases while $\det G_{\mu\nu}$ is finite for case 3). In other words, the light-like NCYM, if they indeed exist, for the former two cases are defined on a zero-size spacetime, or singular spacetime, while the latter is a well-defined usual NCYM. Because of this, at least we have one component of $\alpha'G^{\mu\nu}$ nonvanshing (the same is true in the $SL(2, Z)$ dual). One would say that the underlying theory may not be a field theory. One may wonder that the unitarity condition obtained from a one-loop analysis in [6, 14] is sufficient to show the existence of such light-like NCYM. Further study is needed.

One could have non-singular metric by rescaling the light-cone coordinate x^-. For $1^{(lc)})$, if we rescale $x^- = (\tilde{b}/(\alpha'\sqrt{g_0 g_2}))\tilde{x}^-$, then we get the open string metric $G^{(lc)}_{\mu\nu} = \text{diag}(1/2, 1, -2, 1)$, which is non-singular with respect to the coordinates $(x^+, x^1, \tilde{x}^-, x^3)$. For $2^{(lc)})$, if we rescale $x^- = (\tilde{b}/\alpha')x^-$, we have, with respect to $(x^+, x^1, \tilde{x}^-, x^3)$,

$$
G^{(lc)}_{\mu\nu} = \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\tag{57}
$$

which is also non-singular. Note that the only noncommutativity parameter with respect to (x^+, x^1, x^-, x^3) is Θ^{-1}. Actually, we have the two-point function $\langle x^-(\tau)x^1(0)\rangle = (i/2)\Theta^{-1}\epsilon(\tau)$. The scaling $x^- \sim (1/\alpha')\tilde{x}^-$ implies $\langle \tilde{x}^- x^1 \rangle \sim \alpha'\Theta^{-1}\epsilon(\tau) \to 0$. Since now $\alpha'G^{-1} \to 0$ with respect to $(x^+, x^1, \tilde{x}^-, x^3)$, we therefore end up with light-like OYM[6] for both 1) and 2).

Because we end up with OYM for 1) and 2), their $SL(2, Z)$ dualities still give OYM as discussed at the outset of this section. We will not give the detail here.

We now move on to discuss possible NCOS limit and its $SL(2, Z)$ duality. To have decoupling limit for NCOS, we need to keep $\alpha'G^{\mu\nu}$ and at least Θ^{01} fixed when the limit $E \to E_c$ is taken with E_c the critical field limit. In general, we do not need to send $\alpha' \to 0$ since the open string massive modes are not decoupled from its massless modes. However, it is convenient to choose the $\alpha' \to 0$ limit since we will study the $SL(2, Z)$ duality of the

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[6]We are not sure whether the resulting light-like OYM is physically equivalent to the original theory defined on a zero size 4-dimensional spacetime because the rescaling of x^- is singular.

resulting NCOS which might be a field theory. Now we have a fixed α'_{eff} for the NCOS which is determined by the noncommutative scale.

For $\mathbf{E} \perp \mathbf{B}$, the critical electric field limit is $\tilde{E}^2 \to 1 + \tilde{B}^2$. From the previous discussion for NCYM, we expect that for either fixed $\tilde{B}$ or infinitely large $\tilde{B}$ as $\alpha' \to 0$, we have similar complications here. We will discuss these cases elsewhere. We here focus on the limit $\tilde{B} \to 0$ along with the above critical electric field limit as $\alpha' \to 0$ for NCOS. The relation between the effective open string coupling and the closed string coupling (23) implies $\tilde{E}^2 < 1 + \tilde{B}^2$, so we should have in general $\tilde{E}^2 = 1 + \tilde{B}^2 - (\alpha'/\tilde{b})^\delta$ with $\delta > 0$ and $\tilde{b}$ fixed. With the above discussion, we must also have $\tilde{B}^2 = (\alpha'/\tilde{b}')^\beta$ with $\beta > 0$ and $\tilde{b}'$ fixed. For $\beta > \delta$, the effect of $\mathbf{B}$ simply drops out and we have purely electric field effect which has been discussed before [2, 3] and we will not repeat this case here. The only other case which gives $G_{\mu\nu} \sim \alpha'$ is $\beta = \delta$. We have two cases: a) $\tilde{b} > \tilde{b}'$ and b) $\tilde{b} < \tilde{b}'$. Let us discuss each in order. a) We now have the decoupling limit

$$
\alpha' \to 0, \quad g_0 = \left[\left(\frac{\tilde{b}}{\tilde{b}'} \right)^\delta - 1 \right]^{-1} \left(\frac{\alpha'}{\tilde{b}} \right)^{1-\delta}, \quad g_1 = \left(\frac{\alpha'}{\tilde{b}} \right)^{1-\delta},
$$

$$
g_2 = g_3 = \frac{\alpha'}{\tilde{b}}, \quad g_s = G_s \left(\frac{\tilde{b}}{\alpha'} \right)^{\delta/2}, \quad \tilde{E}^2 = 1 + \left[\left(\frac{\tilde{b}}{\tilde{b}'} \right)^\delta - 1 \right] \left(\frac{\alpha'}{\tilde{b}} \right)^\delta, \tag{58}
$$

with $\tilde{B}$ given above. We then have

$$
G_{\mu\nu} = \frac{\alpha'}{\tilde{b}} \begin{pmatrix} 1 & 0 & -\left[1 - (\tilde{b}'/\tilde{b})^\delta \right]^{-1/2} & 0 \\ 0 & 1 & 0 & 0 \\ -\left[1 - (\tilde{b}'/\tilde{b})^\delta \right] & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Theta^{01} = 2\pi\tilde{b} \left[\left(\frac{\tilde{b}}{\tilde{b}'} \right)^\delta - 1 \right]^{1/2}. \tag{59}
$$

We, therefore, have NCOS with nonvanishing Θ^{01}. We now consider case b)[7]. The decoupling limit for this case remains the same except for the scaling for g_0 which can be obtained by the following replacement:

$$
\left(\frac{\tilde{b}}{\tilde{b}'} \right)^\delta - 1 \to 1 - \left(\frac{\tilde{b}}{\tilde{b}'} \right)^\delta. \tag{60}
$$

Now we have

$$
G_{\mu\nu} = \frac{\alpha'}{\tilde{b}} \begin{pmatrix} -1 & 0 & -\left[(\tilde{b}'/\tilde{b})^\delta - 1 \right]^{-1/2} & 0 \\ 0 & 1 & 0 & 0 \\ -\left[(\tilde{b}'/\tilde{b})^\delta - 1 \right]^{-1/2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{61}
$$

and the nonvanishing noncommutativity parameter Θ^{01} which can be obtained from (59) by the same replacement as above. We again end up with a NCOS.

We denote the $SL(2, Z)$ duality of the above two cases as $a)'$ and $b)'$.

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[7]This case may be equivalent to the one studied in [11].

- a)' We need to consider: 1) irrational χ and 2) rational χ.

  1)**Irrational** χ: Now since $g_s \to \infty$, we have $|c\lambda + d| = |c\chi + d| \neq 0$ and $|cS + d|^2$ remains fixed. So we have

$$\hat{G}_{\mu\nu} = \frac{|cS + d|^2}{|c\chi + d|} G_{\mu\nu}, \quad \hat{G}_s = |cS + d|^2 G_s$$

$$\hat{\Theta}^{01} = \frac{c\chi + d}{|cS + d|^2} \Theta^{01}, \quad \hat{\Theta}^{23} = -\frac{2\pi\tilde{b}c}{G_s|cS + d|^2}, \tag{62}$$

  where $G_{\mu\nu}, G_s$ and Θ^{01} are the open string metric, noncommutativity parameter and open string coupling in a) above. Since $\alpha'\hat{G}^{-1}$ is fixed, so we still have NCOS. This theory is strongly coupled and again the $SL(2, Z)$ duality is not useful.

  2)**Rational** χ: We can now choose $c\chi + d = 0$. Then we have $|c\lambda + d| = |c|/g_s = (|c|/G_s)(\alpha'/\tilde{b})^{\delta/2}$ and $|cS + d|^2 = c^2/G_s^2$ still remains fixed. We then have

$$\hat{G}_{\mu\nu} = \frac{|c|}{G_s} \left(\frac{\alpha'}{\tilde{b}}\right)^{-\delta/2} G_{\mu\nu}, \quad \hat{G}_s = \frac{c^2}{G_s},$$

$$\hat{\Theta}^{23} = -2\pi\tilde{b}G_s/c. \tag{63}$$

  We now have $\alpha'\hat{G}^{-1} \to 0$ and nonvanishing noncommutativity parameter $\hat{\Theta}^{23}$, therefore we end up with a NCYM. This NCYM is weakly coupled if the original NCOS is strongly coupled. Therefore the $SL(2, Z)$ duality is useful.

- b)'. The discussion for this case is basically the same as in a)' above and we do not repeat them here.

4.2   E || B Case

Unlike the previous one, this case is relatively simple since the open string metric is always diagonal and we do not have the same complications as we encountered there. Let us begin with the decoupling limit for NCYM.

For having sensible quantum NCYM, we need to keep the open string metric, the open string coupling and at least one space-space noncommutativity parameter fixed as $\alpha' \to 0$. For simplicity, we choose $G_{\mu\nu} = \eta_{\mu\nu} = (-1, 1, 1, 1)$. From (33), we have $\tilde{E}^2 \leq 1$. We then have the following decoupling limit:

$$\alpha' \to 0, \quad \tilde{B} = \frac{\tilde{b}}{\alpha'}, \quad g_2 = \left(\frac{\tilde{b}}{\alpha'}\right)^2, \quad g_1(1 - \tilde{E}^2) = 1, \quad g_s = \frac{G_s}{\sqrt{1 - \tilde{E}^2}} \frac{\alpha'}{\tilde{b}}. \tag{64}$$

The only nonvanishing noncommutativity parameter is

$$\Theta^{23} = -2\pi\tilde{b}. \tag{65}$$

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In the above, we have not specified how $\tilde{E}$ scales. It appears that the resulting NCYM does not require this as long as $\tilde{E}^2 \leq 1$. However, the scaling behavior of this parameter has great impact on its $SL(2, Z)$ dual description. This dual description may have a small coupling, therefore a good one, in the case when the open string coupling G_s is large. For this purpose, let us consider the following three cases which correspond to those studied in [12]:

- a) $\tilde{E}$ is fixed but it equals neither 0 nor unity[8].

- b) $\tilde{E} = 1 - (\alpha'/\tilde{b}')^{\delta}/2$ with $\delta > 0$.

- c) $\tilde{E} = (\alpha'/\tilde{b}')^{\beta}$, with $\beta > 0$.

We would like to point out that the electric field in b) becomes critical but does not have effect on NCYM.

Let us study each of the above in the $SL(2, Z)$ dual description.

Case a): Using the decoupling limit in (64), we have $|c\lambda + d| = c/g_s = c\tilde{b}(1 - \tilde{E}^2)^{1/2}/(\alpha'G_s)$ and $|cS + d| = c\tilde{B}\tilde{E}/G_s = c\tilde{b}\tilde{E}/(\alpha'G_s)$. Using these we have

$$\hat{G}_{\mu\nu} \sim \eta_{\mu\nu}/\alpha', \quad \hat{\Theta}^{01} \sim \alpha'^2, \quad \hat{\Theta}^{23} \sim \alpha' \quad \hat{G}_s \sim 1/\alpha'^2. \tag{66}$$

Since $\alpha'\hat{G}^{\mu\nu} \sim \eta^{\mu\nu}\alpha'^2 \to 0$, we still have a field theory but defined in a commutative geometry. However, this theory is bad since it has an infinitely large open string coupling and a singular metric. Even if we rescale the coordinates to have a finite metric but we cannot change the open string coupling. So we cannot turn this theory to a well-defined one. We here reach the same conclusion as in [12] regardless of the fact that χ is rational or not.

Case b): This case is not much different from case a). Even though the scaling of the open string metric depends on whether χ is rational or not, it always blows up as $\alpha' \to 0$. So we still end up with a field theory which is not well-defined since the open string coupling blows up in the same way as in case a). The noncommutativity parameters scale as

$$\hat{\Theta}^{01} \sim \alpha'^{2+\delta}, \quad \hat{\Theta}^{23} \sim \alpha'. \tag{67}$$

Case c): From our experience in [12] on S-duality, we expect that this is the case for which we expect to have NCOS. We now have $g_s = \alpha'G_s/\tilde{b}$ and $|c\lambda + d| = c/g_s = c\tilde{b}/(\alpha'G_s)$. We have three sub-cases to consider: $0 < \beta < 1$, $\beta = 1$ and $\beta > 1$. For $0 < \beta < 1$, we reach the same conclusion as in case a) and b) above, i.e., we end up with a field theory which is not well-defined because of the infinitely large open string coupling. This subcase has also been studied in [12] on S-dual rather than on $SL(2, Z)$ dual. The conclusion remains the same and we will skip the details. We now focus on $\beta = 1$ and $\beta > 1$ subcases. For $\beta = 1$,

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[8] $\tilde{E} = 0$ corresponds to zero electric field which is not our interest here. $\tilde{E} = 1$ gives a singular open string metric and the NCYM is no longer $1 + 3$ dimensional which is not our interest here, either. So we exclude these two cases here.

$|cS + d|^2 = [(c\chi + d) + \tilde{b}/(\tilde{b}'G_s)]^2 + c^2/G_s^2$ is fixed and we have

$$\hat{G}_{\mu\nu} = \frac{\alpha'}{\tilde{b}} \frac{G_s}{|c|} |cS + d|^2 \eta_{\mu\nu}, \quad \hat{G}_s = G_s |cS + d|^2,$$

$$\hat{\Theta}^{01} = -\frac{2\pi\tilde{b}}{G_s |cS + d|^2}, \quad \hat{\Theta}^{23} = -\frac{2\pi\tilde{b}}{|cS + d|^2} \left[(d + c\chi) + \frac{c\tilde{b}}{G_s \tilde{b}'} \right]. \tag{68}$$

The above implies that $\alpha'\hat{G}^{\mu\nu}$ is fixed. We therefore have NCOS rather than NCYM. In other words, the $SL(2, Z)$ dual of NCYM for $\beta = 1$ gives a NCOS whether χ is rational or not. This is due to $g_s \to 0$. However, whether χ is rational or not is important in determining the usefulness of the $SL(2, Z)$ duality. Our primary purpose is to find a weakly coupled theory by $SL(2, Z)$ duality when the open string coupling for NCYM is large. When χ is irrational, we map a strongly coupled theory (NCYM) to another strongly coupled theory (NCOS) by $SL(2, Z)$ duality which can be examined from the relation between two open string couplings given in (68). So $SL(2, Z)$ duality is not particularly useful in this case. However, when χ is rational, we can always choose $c\chi + d = 0$ through $SL(2, Z)$ duality. Then we can map a strongly coupled theory (NCYM) to a physically equivalent and weakly coupled theory (NCOS). So only for rational χ, the S-duality is useful.

For $\beta > 1$, we continue to have $|c\lambda + d| = |c|/g_s = |c|\tilde{b}/(\alpha'G_s)$ but now $|cS + d|^2 = (c\chi + d)^2 + c^2/G_s^2$. We then have

$$\hat{G}_{\mu\nu} = \frac{\alpha'}{\tilde{b}} \frac{G_s}{|c|} |cS + d|^2 \eta_{\mu\nu}, \quad \hat{G}_s = G_s |cS + d|^2,$$

$$\hat{\Theta}^{01} = -\frac{2\pi\tilde{b}}{G_s |cS + d|^2}, \quad \hat{\Theta}^{23} = -\frac{2\pi(c\chi + d)}{|cS + d|^2}. \tag{69}$$

We have again NCOS since $\alpha'\hat{G}^{\mu\nu} \sim \eta^{\mu\nu}$. Only for rational χ, a strongly coupled NCYM can be mapped to a weakly coupled NCOS by $SL(2, Z)$ duality since we can choose $c\chi + d = 0$. Once such a choice is made, we have $\hat{\Theta}^{23} = 0$. This case is not different from the one with $\tilde{E} = 0$. However, when χ is irrational, we end up not only with a strongly coupled theory but also with nonvanishing $\hat{\Theta}^{23}$ even if we start with $\tilde{E} = 0$.

Let us now discuss the $SL(2, Z)$ duality of NCOS. To have NCOS, we need $\alpha'G^{\mu\nu}$ and at least Θ^{01} to be fixed when the critical electric field limit $\tilde{E} \to 1$ is taken. Unlike in the field theory limit, we do not need to take $\alpha' \to 0$ since we do not require the open string massive modes to decouple from its massless ones. Our purpose here is to study the $SL(2, Z)$ dual of NCOS which might be a NCYM. For this reason, it is convenient to set $\alpha' \to 0$ for NCOS such that we can easily discuss its $SL(2, Z)$ dual which has the possibility of NCYM. In doing so, the effective open string α'_{eff} for the NCOS is still fixed and is determined by the noncommutativity parameter or scale. Since we require $\alpha'G^{\mu\nu}$ to be fixed as $\alpha' \to 0$, so we can set for simplicity

$$G_{\mu\nu} = \frac{\alpha'}{\tilde{b}} \eta_{\mu\nu}. \tag{70}$$

From eqs.(30)-(33), we have the following decoupling limit:

$$\alpha' \to 0, \quad \tilde{E} = 1 - \frac{1}{2}\left(\frac{\alpha'}{\tilde{b}'}\right)^\delta, \quad g_1 = \frac{\tilde{b}'}{\tilde{b}}\left(\frac{\alpha'}{\tilde{b}'}\right)^{1-\delta},$$

$$g_2(1+\tilde{B}^2) = \frac{a'}{\tilde{b}}, \quad g_s = \left(\frac{\tilde{b}'}{\alpha'}\right)^{\delta/2}\frac{G_s}{(1+\tilde{B}^2)^{1/2}}, \tag{71}$$

with $\delta > 0$. From the above and (32), we have

$$\Theta^{01} = 2\pi\tilde{b}, \quad \Theta^{23} = -2\pi\tilde{b}\tilde{B}. \tag{72}$$

In the above, we have not yet specified how $\tilde{B}$ scales. In general we can set $\tilde{B} = h(\alpha'/\tilde{b}')^\beta$ with h fixed and $\beta \geq 0$. For $\beta = 0$, Θ^{23} is fixed while it vanishes for $\beta > 0$. Using this decoupling limit, we try to find the underlying theory after $SL(2,Z)$ duality. Whether χ is rational or not is crucial for the conclusion. So we discuss them separately in the following.

Irrational χ: In this case we have $c\chi+d \neq 0$. Now $|c\lambda+d| = |c\chi+d|$ since $g_s \to \infty$. For $\beta = 0$, $|cS+d|^2 = [c(\chi+h/G_s)+d]^2+c^2/G_s^2$. Whereas for $\beta > 0$, $|cS+d| = (c\chi+d)^2+c^2/G_s^2$. So for $\beta \geq 0$, $|cS+d|$ is fixed. From (36), (37) and (38), we have

$$\hat{G}_{\mu\nu} = \frac{\alpha'}{\tilde{b}}\frac{|cS+d|^2}{|c\chi+d|}\eta_{\mu\nu}, \quad \hat{G}_s = |cS+d|G_s,$$

$$\hat{\Theta}^{01} = \frac{(c\chi+d)}{|cS+d|^2}\Theta^{01}, \quad \hat{\Theta}^{23} = \frac{(c\chi+d)\Theta^{23} - 2\pi c\tilde{b}(1+\tilde{B}^2)/G_s}{|cS+d|^2}. \tag{73}$$

The scaling of the metric $\hat{G}_{\mu\nu}$ tells that we end up actually with a NCOS rather than a NCYM for irrational χ. This has been given first in [13]. Notice that we now have nonvanishing $\hat{\Theta}^{23}$ even if we begin with $\Theta^{23} = 0$. However, we map a strongly coupled NCOS to another strongly coupled NCOS. Therefore, for irrational χ, the $SL(2,Z)$ duality is not that useful. The interesting point in this case is that we can use it to reduce or to increase the space-space noncommutative directions (since we can get a vanishing Θ^{23} from a nonvanishing $\hat{\Theta}^{23}$ or vice-versa).

Rational χ: Now we can always choose $c\chi+d = 0$. Then $|c\lambda+d| = c/g_s = (\alpha'/\tilde{b}')^{\delta/2}c(1+\tilde{B}^2)^{1/2}/G_s$. Again $|cS+d|^2$ remains fixed. So we have

$$\hat{G}_{\mu\nu} \sim \alpha'^{1-\delta/2}\eta_{\mu\nu}, \quad \hat{\Theta}^{01} = 0, \quad \hat{\Theta}^{23} = -\frac{2\pi c\tilde{b}(1+\tilde{B}^2)}{G_s|cS+d|}, \tag{74}$$

and the open string coupling $\hat{G}_s = c^2(1+h^2)/G_s$ for $\beta = 0$ and $\hat{G}_s = c^2/G_s$ for $\beta > 0$.

Since $\alpha'\hat{G}^{-1} \sim \alpha'^{\delta/2} \to 0$, we therefore end up with a NCYM with noncommutative space-space directions. This has also been studied in [13]. So now a strongly coupled NCOS is physically equivalent to a weakly coupled NCYM. In this case, the $SL(2,Z)$ is really useful and now a $SL(2,Z)$ is not much different from a simple S-duality as studied in [12]. A space-time noncommutativity is also transformed to a space-space one.

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5 Conclusion

To conclude, we have discussed in this paper various decoupling limits for noncommutative open string/Yang-Mills theory in four-dimensions and their $SL(2, Z)$ duality for both $\mathbf{E} \perp \mathbf{B}$ and $\mathbf{E}||\mathbf{B}$ cases. Since $SL(2, Z)$ is a non-perturbative quantum symmetry of type IIB string theory, we often use this symmetry to find a physically equivalent and yet weakly coupled theory if the original theory is strongly coupled. However, our study indicates that if the RR scalar in one theory is irrational, the $SL(2, Z)$ does not help much and the $SL(2, Z)$ dual is always strongly coupled. So when we say that we can use S-duality or in general $SL(2, Z)$ duality to transform a strongly coupled theory to a weakly coupled one, one must understand that this can be done only for rational χ. Since χ is determined by the underlying (most likely non-perturbative) vacuum, whether χ is rational or not is a rather non-trivial question. We cannot answer this until we understand the non-perturbative type IIB theory completely.

We also find that $SL(2, Z)$ symmetry can be used to increase or decrease the number of noncommutative directions but it seems that we cannot turn an OYM to a NCYM/NCOS through this symmetry. We also find that the interplay of electric and magnetic fields are important in controlling the number of noncommutative directions. We show that the $SL(2, Z)$ duality of NCYM can be an ordinary theory which is not well-defined or a NCOS regardless of whether χ is rational or not. But only for rational χ, the NCOS can be weakly coupled if the original NCYM is strongly coupled. Also when the original NCOS is strongly coupled the $SL(2, Z)$ duality is either another strongly coupled NCOS if χ is irrational or a weakly coupled NCYM if χ is rational. Some of the critical electric field limit for $\mathbf{E} \perp \mathbf{B}$ are particularly interesting. Whether we have decoupled light-like NCYM or light-like NCOS or other kinds of OS or light-like OYM is still not clear. Further study is needed.

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